The Definition of Product of Inertia

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Abstract

The term "product of inertia" is an ambiguous term in the engineering community. The ambiguity arises because of the lack of a standard sign convention for describing certain terms associated with the mass distribution of a body. The difference between the two competing standards is simply a negative sign, and the reason for this difference is explained below.

1 Introduction

The concept of *mass moment of inertia* is usually first introduced in undergraduate physics and engineering courses. Quite often it is explained as the rotational counterpart to mass. This analogy is compeling because Newton's equation which governs the *translational* motion of a rigid body B in a plane perpendicular to the unit vector $\mathbf{z}$ is

$$\mathbf{F} \;=\; m\,\mathbf{a} \tag{1}$$

where $\mathbf{F}$ is the resultant force acting on B; m is the mass of B; and $\mathbf{a}$ is the acceleration of B_o, the center of mass of B. The *rotational* counterpart to equation (1) is a simplified form of Euler's dynamical equation which is suitable for planar analysis, namely

$$\mathbf{T} \;=\; I_{zz}\,\boldsymbol{\alpha} \tag{2}$$

where $\mathbf{T}$ is the $\mathbf{z}$ component of the moment of all forces about B_o; I_{zz} is the mass moment of inertia about the line passing through B_o and parallel to $\mathbf{z}$; and $\boldsymbol{\alpha}$ is the angular acceleration of B.

Although this analogy is "intuitive", it fails to have a three-dimensional counterpart. Newton's law which governs the *translational* motion of B in three-dimensional space is simply equation (1). However, Euler's law for three-dimensional *rotational* motions of B is more elaborate, namely

$$\mathbf{T} \;=\; \underline{\mathbf{I}}\cdot\boldsymbol{\alpha} \;+\; \boldsymbol{\omega}\times\underline{\mathbf{I}}\cdot\boldsymbol{\omega} \tag{3}$$

where $\mathbf{T}$ is the moment of all forces about B_o; $\underline{\mathbf{I}}$ is the central inertia dyadic of B (we will return to this momentarily); and $\boldsymbol{\omega}$ is the angular velocity of B. The compact represention of equation (3) can be misleading. When it is written out in scalar form, it is much longer. For example, when the

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vectors $\mathbf{T}$ and $\boldsymbol{\omega}$ appearing in equation (3) are expressed in terms of the orthogonal unit vectors $\mathbf{b}_\mathrm{x}$, $\mathbf{b}_\mathrm{y}$, $\mathbf{b}_\mathrm{z}$ fixed in B as

$$\begin{aligned} \mathbf{T} &= T_x\mathbf{b}_\mathrm{x} + T_y\mathbf{b}_\mathrm{y} + T_z\mathbf{b}_\mathrm{z} \\ \boldsymbol{\omega} &= \omega_x\mathbf{b}_\mathrm{x} + \omega_y\mathbf{b}_\mathrm{y} + \omega_z\mathbf{b}_\mathrm{z} \end{aligned} \tag{4}$$

then the scalar equations of motion can be expressed in terms of B's moments of inertia I_{xx}, I_{yy}, I_{zz}, and B's products of inertia I_{xy}, I_{xz}, I_{yz} as[1]

$$T_x = I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z + \omega_y(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) \tag{5}$$
$$T_y = I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z + \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) - \omega_x(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z) \tag{6}$$
$$T_z = I_{xz}\dot{\omega}_x + I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z + \omega_x(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) \tag{7}$$

which, when $I_{xy} = I_{xz} = I_{yz} = 0$, can be reduced to

$$T_x = I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy}) \tag{8}$$
$$T_y = I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz}) \tag{9}$$
$$T_z = I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx}) \tag{10}$$

The symbol $\underline{\mathbf{I}}$ appearing in equation (3) shows up in many other other useful dynamical relationships. For example, the *angular momentum* of B is

$$\mathbf{H} = \underline{\mathbf{I}} \cdot \boldsymbol{\omega} \tag{11}$$

and the kinetic energy of B is

$$K = \frac{1}{2}\boldsymbol{\omega} \cdot \underline{\mathbf{I}} \cdot \boldsymbol{\omega} \tag{12}$$

Before continuing with an involved description of a *central inertia dyadic*, it is helpful to have a clear understanding of dyadics. This understanding is most clearly communicated by first discussing the relationship between vectors and column matrices and then focusing attention on the relationship between dyadics and $3x3$ matrices.

2 Vectors, Dyadics and Matrices

A vector is a quantity with a magnitude and an associated direction. A vector can be *expressed* in a variety of ways. For example the vector $\mathbf{v}$ can be expressed in terms of the orthogonal unit vectors $\mathbf{b}_\mathrm{x}$, $\mathbf{b}_\mathrm{y}$, $\mathbf{b}_\mathrm{z}$ as

$$\mathbf{v} = \beta_x\,\mathbf{b}_\mathrm{x} + \beta_y\,\mathbf{b}_\mathrm{y} + \beta_z\,\mathbf{b}_\mathrm{z} \tag{13}$$

which can also be expressed as

$$\mathbf{v} = \begin{bmatrix} \mathbf{b}_\mathrm{x} & \mathbf{b}_\mathrm{y} & \mathbf{b}_\mathrm{z} \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} \tag{14}$$

[1]Because of the variety of definitions of product of inertia, there is an ambiguity on the sign in front of I_{xy}, I_{xz}, I_{yz} in equations (5-7)

The 3x1 matrix associated with the vector in equation (14) is

$$v_b = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} \tag{15}$$

The subscript B in equation (15) denotes that the matrix is associated with $\mathbf{b}_\mathrm{x}$, $\mathbf{b}_\mathrm{y}$, and $\mathbf{b}_\mathrm{z}$. This subscript may be omitted when the context is clear.

A dyad is a quantity with magnitude and *two* associated directions. A dyadic is the sum of one or more dyads. A dyadic can be expressed in a variety of ways. For example the dyadic $\underline{\mathbf{I}}$ can be expressed in terms of the orthogonal unit vectors $\mathbf{b}_\mathrm{x}$, $\mathbf{b}_\mathrm{y}$, $\mathbf{b}_\mathrm{z}$ as

$$\begin{aligned} \underline{\mathbf{I}} &= \beta_{xx}\mathbf{b}_\mathrm{x}\mathbf{b}_\mathrm{x} + \beta_{xy}\mathbf{b}_\mathrm{x}\mathbf{b}_\mathrm{y} + \beta_{xz}\mathbf{b}_\mathrm{x}\mathbf{b}_\mathrm{z} \\ &+ \beta_{yx}\mathbf{b}_\mathrm{y}\mathbf{b}_\mathrm{x} + \beta_{yy}\mathbf{b}_\mathrm{y}\mathbf{b}_\mathrm{y} + \beta_{yz}\mathbf{b}_\mathrm{y}\mathbf{b}_\mathrm{z} \\ &+ \beta_{zx}\mathbf{b}_\mathrm{z}\mathbf{b}_\mathrm{x} + \beta_{zy}\mathbf{b}_\mathrm{z}\mathbf{b}_\mathrm{y} + \beta_{zz}\mathbf{b}_\mathrm{z}\mathbf{b}_\mathrm{z} \end{aligned} \tag{16}$$

which can also be written as

$$\underline{\mathbf{I}} = \begin{bmatrix} \mathbf{b}_\mathrm{x} & \mathbf{b}_\mathrm{y} & \mathbf{b}_\mathrm{z} \end{bmatrix} \begin{bmatrix} \beta_{xx} & \beta_{xy} & \beta_{xz} \\ \beta_{yx} & \beta_{yy} & \beta_{yz} \\ \beta_{zx} & \beta_{zy} & \beta_{zz} \end{bmatrix} \begin{bmatrix} \mathbf{b}_\mathrm{x} \\ \mathbf{b}_\mathrm{y} \\ \mathbf{b}_\mathrm{z} \end{bmatrix} \tag{17}$$

The 3x3 matrix associated with the dyadic in equation (17) is

$$I_b = \begin{bmatrix} \beta_{xx} & \beta_{xy} & \beta_{xz} \\ \beta_{yx} & \beta_{yy} & \beta_{yz} \\ \beta_{zx} & \beta_{zy} & \beta_{zz} \end{bmatrix} \tag{18}$$

The subscript B in equation (18) denotes that the matrix is associated with $\mathbf{b}_\mathrm{x}$, $\mathbf{b}_\mathrm{y}$, and $\mathbf{b}_\mathrm{z}$. This subscript may be omitted when the context is clear or when the matrix is the identity matrix (which implies that the dyadic is a unit dyadic).

3 Inertia Properties

The integral that must be perfomed to calculate the central inertia dyadic of a rigid body B, is

$$\underline{\mathbf{I}} = \int (\underline{\mathbf{U}}\mathbf{p}^2 - \mathbf{pp})\, dm \tag{19}$$

where $\underline{\mathbf{U}}$ is the unit dyadic; $\mathbf{p}^/$ is the position vector from B_o, the mass center of B, to an arbitrary point on B; and dm is the mass of a differential element of B. $\underline{\mathbf{I}}$, the dyadic which results from performing this integral is the quantity which is used in connection with equations (3) - (12).

3.1 Moments of Inertia

The *moments of inertia* are usually designated I_{xx}, I_{yy}, and I_{zz} and each moment of inertia is associated with a line. For example, I_{xx} is associated with a line passing through B_o and parallel to $\mathbf{b}_\mathrm{x}$. The moments of inertia can be found in a variety of ways. One way to find I_{xx} is to form the inertia dyadic as specified in equation (19) and then perform the following dot-products with $\mathbf{b}_{\mathrm{x}}$:

$$I_{xx} \;=\; \mathbf{b}_{\mathrm{x}} \cdot \underline{\mathbf{I}} \cdot \mathbf{b}_{\mathrm{x}} \tag{20}$$

A second way to find I_{xx} is to form the inertia dyadic as specified in equation (19), form the associated inertia matrix like the one in equation (18), and then note that

$$I_{xx} \underset{(18)}{=} \beta_{xx} \tag{21}$$

A third way to find I_{xx} is to express $\mathbf{p}^{/}$ as

$$\mathbf{p}^{/} \;=\; x\,\mathbf{b}_{\mathrm{x}} \;+\; y\,\mathbf{b}_{\mathrm{y}} \;+\; z\,\mathbf{b}_{\mathrm{z}} \tag{22}$$

and then calculate I_{xx} directly by performing the following integral

$$I_{xx} \underset{(19,22)}{=} \int (y^2 + z^2)\, dm \tag{23}$$

3.2 Products of Inertia

The *products of inertia* are usually designated I_{xy}, I_{yz}, and I_{zx} and each product of inertia is associated with two lines. For example, I_{yz} is associated with a line passing through B_o and parallel to $\mathbf{b}_{\mathrm{y}}$ and a second line passing through B_o and parallel to $\mathbf{b}_{\mathrm{z}}$. The products of inertia can be found in a variety of ways. One way to find I_{yz} is to form the inertia dyadic as specified in equation (19) and then perform the following dot-products with $\mathbf{b}_{\mathrm{y}}$ and $\mathbf{b}_{\mathrm{z}}$.

$$I_{yz} \;=\; \mathbf{b}_{\mathrm{y}} \cdot \underline{\mathbf{I}} \cdot \mathbf{b}_{\mathrm{z}} \tag{24}$$

A second way to find I_{yz} is to form the inertia dyadic as specified in equation (19), form the associated inertia matrix like the one in equation (18), and then note that

$$I_{yz} \underset{(18)}{=} \beta_{yz} \tag{25}$$

A third way to calculate I_{yz} is to express $\mathbf{p}^{/}$ as was done in equation (22), and then calculate I_{yz} directly by performing the following integral

$$I_{yz} \underset{(19,22)}{=} -\int y\; z\; dm \tag{26}$$

4 Differences in opinion

The confusion surrounding "products of inertia" in the engineering community is further exasperated because there are two distinct quantities. First there is the term "product of inertia" and then there is the symbol I_{yz}, which may or may not directly represent a product of inertia. There are four distinct opinions and they are listed below. Along with these opinions are a list of various proponents of each convention.

- Some engineering authors define both the symbol I_{yz} and the term "product of inertia" to mean the integral in equation (26) *with* the negative sign. The proponents of this convention include [1, pg. 11] and [2, pg. 172]. [3], [4, pg. 303], [5, pg. 220], [6, pg. 172], [7, pg. 62], [8], [9, pg. 237], [10, pg. 88], [11, pg. 199]
- Some engineering authors find the negative sign in equation (26) to be objectionable, so they define both the symbol I_{yz} and the term "product of inertia" as the integral in equation (26) *without* the negative sign. The off-diagonal terms of the inertia matrix are then negated before being used. The proponents of this second convention include [12, pg. 1017], [13, pg. 719], [14], [15], [16], and [17, pg. 129].
- A third group of engineering authors define the term "product of inertia" as the integral in equation (26) *without* the negative sign, but define the symbol I_{yz} with the negative sign. The proponents of this third convention include [18], [19, pg. 418], and [20].
- Lastly, some authors avoid the term "product of inertia" altogether and define the symbol I_{yz} as the integral in equation (26) *with* the negative sign, The proponents of this fourth option include [21] and [22].

To make matters worse, there is no agreement among vendors of commerically available multi-body dynamics programs. For example, VISUAL NASTRAN™ (formerly called Working Model) and AUTOLEV™use the integral in equation (26) *with* the negative sign, but ADAMS™, uses a *positive* summation.

As a result of the differing definitions, an engineer should be precise when describing what is meant by the term "product of inertia" and their associated symbols.

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