Determination of Spring Constants for Modeling Flexible Beams

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Abstract

At times, it is convenient to model a flexible beam as a set of rigid bodies connected by linear springs. The modeling of a beam as a set of rigid bodies connected flexibly rather than as is done with finite element codes, namely, modeling a beam as a set of flexible bodies connected rigidly, has several advantages. It provides for selective modeling of flexible objects in a rigid body code, permits an analyst to easily design a control system which accounts for flexibility of system components, and incorporates certain dynamical nonlinear effects, for example centrifugal stiffening, in the simulation of flexible systems. In modeling the beam in this manner, it is necessary to determine the values of these spring constants. This paper provides a method for this determination.

1 Introduction

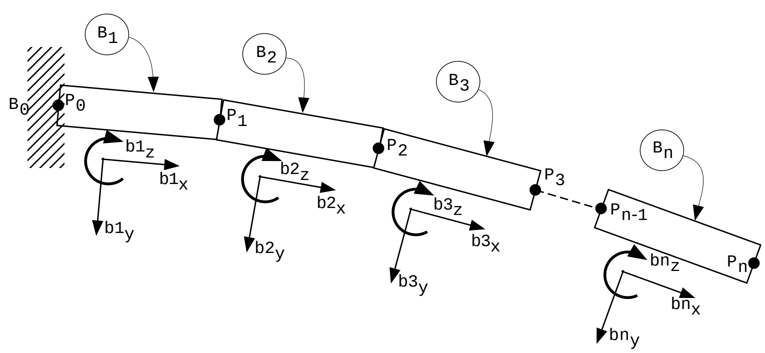

Figure 1: Discretization of Flexible Beam

Figure 1 is a schematic representation of a base B_0 attached to a flexible beam which has been discretized into n rigid bodies B_i, (i=1, ..., n). Body B_i is connected to body B_{i+1} at point P_i by means of one linear *torsional spring*, two linear *bending springs*, and one linear *extensional*

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spring. The torsional spring restricts twisting of the beam about the line connecting P_i to P_{i+1}, the bending springs restrict bending of the beam in directions transverse to the line connecting P_i to P_{i+1}, and the extensional spring restricts elongation of the beam. The spring constants associated with torsion, bending, and extension are K_i^θ, K_i^ϕ, K_i^ψ, and K_i^x, respectively. The length of each beam element when the beam is undeformed is denoted L, and the stretch of the extensional spring is denoted x_i. The elastic modulus of B_i is denoted E, and its area moment of inertia is I.

To characterize the orientation of B_i in B_{i-1}, a right-handed set of mutually perpendicular unit vectors $\mathbf{b_{i}}_\mathrm{x}$, $\mathbf{b_{i}}_\mathrm{y}$, $\mathbf{b_{i}}_\mathrm{z}$, is fixed in B_i with $\mathbf{b_{i}}_\mathrm{x}$ directed from P_i to P_{i+1}. The orientation of B_i in B_{i-1} is found by first setting $\mathbf{b_{i}}_\mathrm{x} = \mathbf{b_{i-1}}_\mathrm{x}$, $\mathbf{b_{i}}_\mathrm{y} = \mathbf{b_{i-1}}_\mathrm{y}$, and $\mathbf{b_{i}}_\mathrm{z} = \mathbf{b_{i-1}}_\mathrm{z}$, and then subjecting B_i to the body-fixed rotation sequence described in magnitude and direction by $\theta_i \mathbf{b_{i}}_\mathrm{x}$, $\phi_i \mathbf{b_{i}}_\mathrm{y}$, and $\psi_i \mathbf{b_{i}}_\mathrm{z}$. In its undeformed configuration, $\mathbf{b_{i}}_\mathrm{y}$ (i=1, ..., n) is pointing vertically downward, as shown in Figure 1.

It is *assumed* that the potential function V_i of the set of springs connecting B_i to B_{i-1} can be written as

$$V_i = \frac{1}{2}K_i^\theta\ \theta_i^2 + \frac{1}{2}K_i^\phi\ \phi_i^2 + \frac{1}{2}K_i^\psi\ \psi_i^2 + \frac{1}{2}K_i^x\ x_i^2 \qquad (i\text{=}1,\ ...,\ n) \tag{1}$$

The contribution of V_i to F_r , the generalized active forces[1], is given by

$$(F_r)_{V_i} = -\frac{\partial V}{\partial q_r} \qquad (r\text{=}1,\ ...,\ 4n) \tag{2}$$

where q_r represents any of θ_i, ϕ_i, ψ_i, or x_i (i=1, ..., n). When the beam is at rest in a Newtonian reference frame and when the generalized coordinates θ_i, ϕ_i, ψ_i, x_i (i=1, ..., n) are independent of each other, then the set of equations which govern the static configuration of the beam are given by [1, p. 179],

$$F_r = 0 \qquad (r\text{=}1,\ ...,\ 4n) \tag{3}$$

2 Extensional Spring Constants

If the only load applied to the beam is a force of magnitude R applied horizontally to point P_n then equations (1), (2), and (3) can be used to show that the non-zero generalized active forces are given by

$$-K_i^x\ x_i\ +\ R \underset{(1,2,3)}{=} 0 \qquad (i\text{=}1,\ ...,\ n) \tag{4}$$

The solution of equation (4) for K_i^x is

$$K_i^x \underset{(4)}{=} R\ /\ x_i \qquad (i\text{=}1,\ ...,\ n) \tag{5}$$

[1]Generalized active forces are used in various dynamic's formulation methods, e.g, Kane's method (see [1]) and Lagrange's method

As will become relevant momentarily, x_i, the extension of B_i, can be expressed in terms of $\delta[iL]$ and $\delta[(i+1)L]$, the X-displacements of P_i and P_{i+1}, as

$$x_i = \delta[(i+1)L] - \delta[iL] \qquad (i=1, ..., n) \tag{6}$$

Using a solution obtained from elasticity theory [2, p. 74 Eq. 3], one may express $\delta[iL]$ in terms of E, Young's elastic modulus, and A, the cross sectional area, as

$$\delta[iL] = \frac{RiL}{AE} \qquad (i=1, ..., n) \tag{7}$$

Substituting from equation (7) into equation (6), one arrives at

$$x_i \underset{(6,7)}{=} \frac{RL}{AE} \qquad (i=1, ..., n) \tag{8}$$

Lastly, substitution of equation (8) into equation (5) produces

$$K_i^x \underset{(5,8)}{=} \frac{AE}{L} \qquad (i=1, ..., n) \tag{9}$$

3 Torsional Spring Constants

If the only load applied to the beam is a torque $T_n \mathbf{b_{nx}}$ applied to body n, then equations (1), (2), and (3) can be used to show that the non-zero generalized active forces are given by

$$-K_i^\theta \, \theta_i \; + \; T_n \underset{(1,2,3)}{=} 0 \qquad (i=1, ..., n) \tag{10}$$

The solution of equation (10) for K_i^θ is

$$K_i^\theta \underset{(10)}{=} T_n \; / \; \theta_i \qquad (i=1, ..., n) \tag{11}$$

As will become relevant momentarily, θ_i, the "relative twist" of B_i, can be expressed in terms of α_i, the "absolute twist" of the beam at the midpoint of B_i, and α_{i-1}, the "absolute twist" of the beam at the midpoint of B_{i-1}, as

$$\theta_1 = \alpha_1 \tag{12}$$

$$\theta_i = \alpha_i - \alpha_{i-1} \qquad (i=2, ..., n) \tag{13}$$

Using a solution obtained from elasticity theory [2, p. 287 Eq. 1], one may express α_i in terms of G, the shear modulus, and J, the polar moment of inertia, as

$$\alpha_i = \frac{T_n(i-0.5)L}{JG} \qquad (i=1, ..., n) \tag{14}$$

Substituting from equation (14) into equations (12) and (13), one arrives at

$$\theta_1 \underset{(12,14)}{=} \frac{T_n L}{2JG} \tag{15}$$

$$\theta_i \underset{(13,14)}{=} \frac{T_n L}{JG} \qquad (i{=}2,\ ...,\ n) \tag{16}$$

Lastly, substitution of equations (15) and (16) into equation (11) produces

$$K_1^\theta \underset{(11,15)}{=} \frac{2JG}{L} \tag{17}$$

$$K_i^\theta \underset{(11,16)}{=} \frac{JG}{L} \qquad (i{=}2,\ ...,\ n) \tag{18}$$

4 Bending Spring Constants

If the only applied load on the beam is a force of magnitude R directed vertically downward and applied to point P_n, then equations (1), (2), and (3) can be used to show that the non-zero generalized active forces are given by

$$-K_i^\phi\ \phi_i\ +\ R\sum_{r=i}^{n}(L+x_r)\cos(\phi_1+\phi_2+\ldots\phi_r) \underset{(1,2,3)}{=} 0 \qquad (i{=}1,\ ...,\ n) \tag{19}$$

Approximating $L + x_r$ as L and linearizing equation (19) in ϕ_i (i=1, ..., n) (deflections are small) results in

$$-K_i^\phi\ \phi_i\ +\ RL(n+1-i) \underset{(19)}{=} 0 \qquad (i{=}1,\ ...,\ n) \tag{20}$$

The solution of equation (20) for K_i^ϕ is

$$K_i^\phi \underset{(20)}{=} RL(n+1-i)\ /\ \phi_i \qquad (i{=}1,\ ...,\ n) \tag{21}$$

As will become relevant momentarily, ϕ_i, the "relative bending angle", may be expressed in terms of β_i and β_{i-1}, "absolute bending angles", as

$$\phi_1 = \beta_1 \tag{22}$$

$$\phi_i = \beta_i - \beta_{i-1} \qquad (i{=}2,\ ...,\ n) \tag{23}$$

Using purely geometric considerations, β_i can be related to Y_i, the vertical displacement of the beam at P_i, by

$$\sin\beta_i = \frac{Y_i - Y_{i-1}}{L+x_i} \qquad (i{=}1,\ ...,\ n) \tag{24}$$

Approximating $L + x_i$ as L and linearizing equation (24) in β_i (i=1, ..., n) leads to

$$\beta_i \underset{(24)}{=} \frac{Y_i - Y_{i-1}}{L} \qquad (i{=}1,\ ...,\ n) \tag{25}$$

Using the solution obtained from Euler beam theory for a uniform cantilevered beam with an end load [2, p. 96 Fig. 1a], one may express Y_i in terms of E, Young's elastic modulus, I, the area moment of inertia, and L, the total length of the beam, as

$$Y_i = \frac{R\,L^3\,(3ni^2 - i^3)}{6\,E\,I} \qquad (i{=}1,\ ...,\ n) \tag{26}$$

Substituting from equation (26) into equation (25) results in

$$\beta_i \underset{(25,26)}{=} \frac{R\,L^2}{6\,E\,I}(6in - 3n + 3i - 3i^2 - 1) \tag{27}$$

$$\beta_{i-1} \underset{(25,26)}{=} \frac{R\,L^2}{6\,E\,I}(6in - 9n + 9i - 3i^2 - 7) \tag{28}$$

Substitution of equations (27) and (28) into equations (22) and (23) results in

$$\phi_1 \underset{(22,27)}{=} \frac{R\,L^2}{6\,E\,I}(3n - 1) \tag{29}$$

$$\phi_i \underset{(23,27,28)}{=} \frac{R\,L^2}{6\,E\,I}(6n - 6i + 6) \qquad (i{=}2,\ ...,\ n) \tag{30}$$

Lastly, substitution of equations (29) and (30) into equation (21) produces

$$K_1^\phi \underset{(21,29)}{=} \frac{EI}{L}\frac{6n}{3n-1} \tag{31}$$

$$K_i^\phi \underset{(21,30)}{=} \frac{EI}{L} \qquad (i{=}2,\ ...,\ n) \tag{32}$$

5 Results

It is natural to wonder if the values for the extensional, torsional and bending spring constants work for a wide range problems. One may wonder what happens under different loading conditions or different end conditions. Alternately, one may wonder how well this technique predicts dynamic phenomenon, e.g., natural frequencies. To that end, a variety of calculations were performed on a uniform beam using the symbolic manipulator AUTOLEV [3]. The results of these calculations are recorded below:

1. Bending of ten-element `cantilever` beam with both mid-point and end loads and end-torque: For both small and large loads, the deflections and curvature are visually indistinguishable from those predicted by Euler beam theory.

2. Bending of ten-element `pin-pin` beam with mid-point, 3/4-point, and end torques:
   For small loads, the deflections and curvature match those predicted by Euler beam theory very well. For larger loads, the deflections are *smaller* than those predicted by Euler beam theory. This discrepancy is understood in light of the fact that Euler beam theory predicts the same deformation for both a pin-pin and pin-roller beam.

3. Bending of ten-element `pin-roller` beam with mid-point, 3/4-point, and end torques:
   For both small and large loads, the deflections and curvature are visually indistinguishable from those predicted from Euler beam theory. In light of the results in item 2, it is clear that this technique approximates axial stiffening.

4. Bending of ten-element `cantilever-cantilever` beam with 3/4-point loads:
   For small loads, the deflections and curvature match those predicted by Euler beam theory very well. For larger loads, the deflections are *smaller* than those predicted by Euler beam theory. This discrepancy is understood in light of the fact that Euler beam theory predicts the same deformation for both a `cantilever-cantilever` and `cantilever-cantilever on a roller` beam.

5. Bending of ten-element `cantilever-cantilever on a roller` beam with 3/4-point loads:
   For both small and large loads, the deflections and curvature are visually indistinguishable from those predicted from Euler beam theory. In light of the results in item 4, it is clear that this technique approximates axial stiffening.

6. First three natural bending, torsional, or extensional frequencies of a `cantilever` beam:
   For a three element beam, the error in the smallest natural frequency was less than 2%. For an eight element beam, the error in the three smallest natural frequencies were less than 5%.

References

[1] KANE, THOMAS R. and LEVINSON, DAVID A. *Dynamics: Theory and Applications*, McGraw-Hill, New York, 1985, pp. 40-43.

[2] ROARK, RAYMOND J., and YOUNG, WARREN C., *Formulas for Stress and Strain* Fifth Edition," McGraw-Hill, New York, 1975.

[3] SCHAECHTER, DAVID B., and LEVINSON, DAVID A., "Interactive Computerized Symbolic Dynamics for the Dynamicist," *The Journal of the Astronautical Sciences*, Vol. 36, No. 4, October-December 1988, pp. 365-388.