Kalish/Montague and Jaśkowski Natural Deduction

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The Latex style file natded.sty will produce representations of natural deduction proofs in either Jaśkowski's original style or the modification of that style by Kalish-Montague. But before describing some of the details of the use of the natded.sty (including the use of "guards"), we pause for some relevant historical information.

The history of the formal approach to natural deduction dates from 1934, when two papers appeared simultaneously in different journals written by authors who had no interaction of any type with one another. Stanisław Jaśkowski ("On the Rules of Suppositions in Formal Logic", *Studia Logica*, v. 1) and Gerhard Gentzen ("Untersuchungen über das logische Schließen" ["Investigations into Logical Deduction"], *Mathematische Zeitschrift* v. 39) worked on the same problem – of trying to formally mimic the reasoning of "ordinary mathematicians" who would "make assumptions and see where they lead" and only later conclude with something that was not dependent on those assumptions.

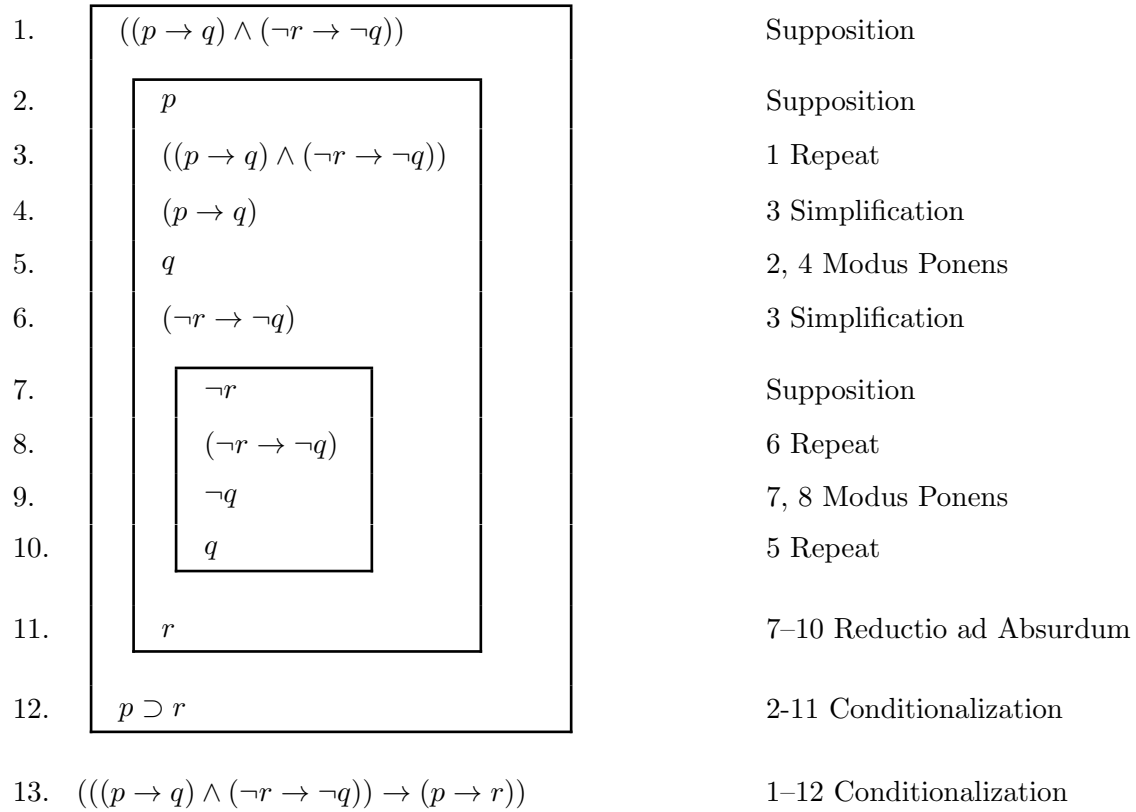
Although they had essentially the same motivation, and arrived at similar but not quite identical conclusions, their methods of representing this sort of reasoning were quite different. Gentzen used a tree format, which can be mimicked using Sam Buss's bussproofs.sty. For example, the proof of the propositional logic theorem $\vdash (((p \to q) \wedge (\neg r \to \neg q)) \to (p \to r))$ takes this form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p \to q) \wedge (\neg r \to \neg q))}\,1}{(\neg r \to \neg q)}\wedge\text{-E} \quad \neg r\,3}{\neg q}\to\text{-E} \quad \dfrac{\dfrac{\dfrac{}{((p \to q) \wedge (\neg r \to \neg q))}\,1}{(p \to q)}\wedge\text{-E} \quad p\,2}{q}\to\text{-E}}{\bot}\bot\text{-I}}{r}\bot\text{-E}\,(3)}{(p \to r)}\to\text{-I}\,(2)}{(((p \to q) \wedge (\neg r \to \neg q)) \to (p \to r))}\to\text{-I}\,(1)$$

A perhaps more familiar style of natural deduction proofs, especially among those who learned their elementary logic in philosophy departments, are the ones usually called "Fitch" representations. For this type of proof representation, there are two Latex packages in common use: Johan Klüwer's fitch.sty and Peter Selinger's fitch.sty. Minor variants of these style packages are available, such as Richard Zach's lplfitch.sty. Here's that same proof using Klüwer's fitch.sty:

1	$((p \to q) \wedge (\neg r \to \neg q))$	
2	$\quad p$	
3	$\quad ((p \to q) \wedge (\neg r \to \neg q))$	1, Reiteration
4	$\quad (p \to q)$	3,$\wedge$E
5	$\quad q$	2,4 $\to$E
6	$\quad (\neg r \to \neg q)$	3,$\wedge$E
7	$\quad\quad \neg r$	
8	$\quad (\neg r \to \neg q)$	6,Reiteration
9	$\quad \neg q$	7,8 $\to$E
10	$\quad q$	5,Reiteration
11	$\quad r$	7–10, $\neg$E
12	$(p \to r)$	2–11, $\to$I
13	$(((p \to q) \wedge (\neg r \to \neg q)))$	1–12,$\to$I

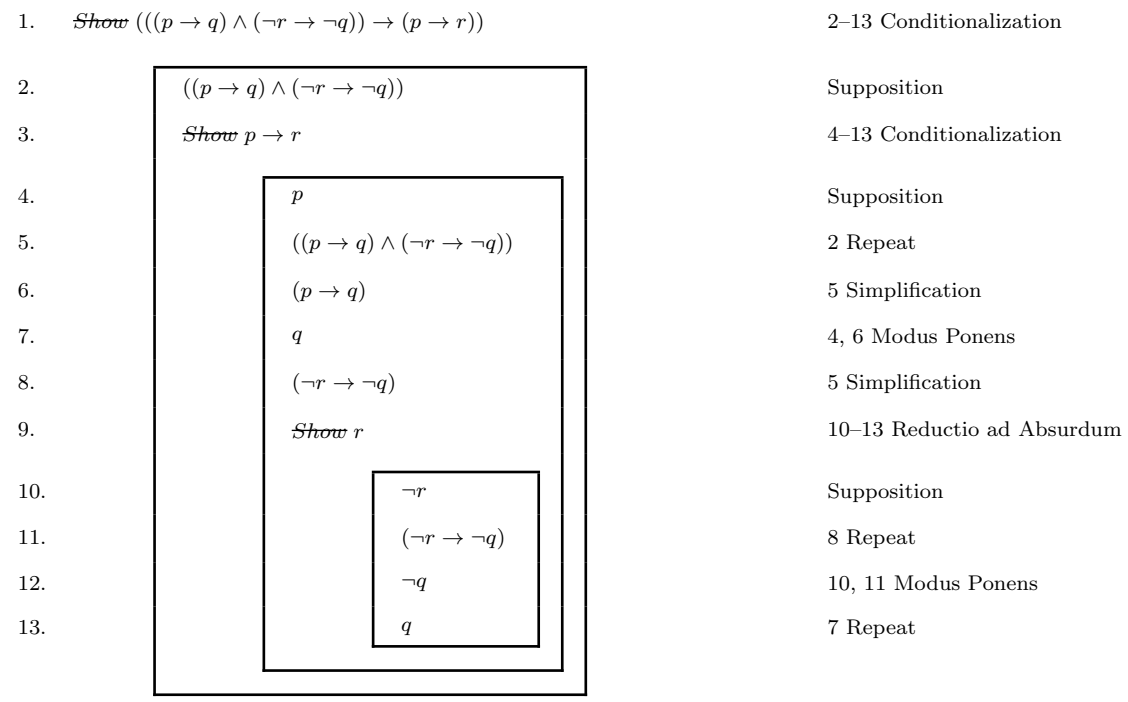
This method is derived from one (of two) methods for natural deduction proof representation described by Jaśkowski (1934). However, this fitch method (from Fredric Fitch (1952) *Symbolic Logic*) is not exactly the way Jaśkowski did his proofs. Here's that same proof in this one of his methods:



It can be seen that what Fitch did was to remove all but the left-side of the boxes (rectangles) that Jaśkowski employed to indicate the new "world of the supposition". And Fitch underlined the assumption or hypothesis or supposition of each such world, which is the first line inside one of Jaśkowski's boxes. Although the difficulty of typesetting these boxes caused Fitch's method to became more common, at least one textbook employed a variant on this method of Jaśkowski's, namely D. Kalish & R. Montague's (1964) *Logic* and the expanded D. Kalish, R. Montague & G. Mar (1980) *Logic*. One noticeable difference is that the Kalish-Montague method placed the conclusion of each one of the boxes at the *beginning* of the subproof, just before the assumption. This was indicated by the word SHOW, so that when engaged in a (sub)proof, one starts by writing the desired conclusion of that (sub)proof, prefixed by this SHOW. And when one legitimately completes that subproof, one "cancels" the SHOW by drawing a line through it, which

indicates that the conclusion can become a part of the next-outer (sub)proof. Here is that same theorem proved in the Kalish-Montague system.

The codes for producing the Jaśkowski and Kalish-Montague proofs are in Listings **??** and **??**, respectively.



Listing 1: LaTeX code for Jaśkowski-style proof

```
\[
\Jproof{
  \cablk{
    \proofline {((p\rightarrow q)\land(\neg r \rightarrow\neg q))}{Supposition}
    \cablk{
      \proofline {p}{Supposition}
      \proofline {((p\rightarrow q)\land(\neg r\rightarrow\neg q))}{1 Repeat}
      \proofline {(p\rightarrow q)}{3 Simplification}
      \proofline {q}{2, 4 Modus Ponens}
      \proofline {(\neg r\rightarrow\neg q)}{3 Simplification}
      \cablk{
        \proofline {\neg r}{Supposition}
        \proofline {(\neg r\rightarrow\neg q)}{6 Repeat}
        \proofline {\neg q}{7, 8 Modus Ponens}
        \proofline {q}{5 Repeat}
      }{
```

```
17          \proofline {r}{7--10 Reductio ad Absurdum}
18        }
19      }{
20        \proofline {p\supset r}{2-11 Conditionalization}
21      }
22    }{
23      \proofline {(((p\rightarrow q)\land(\neg r\rightarrow\neg q)) \rightarrow(p \rightarrow r))}
24    }
25  }
26  \]
```

Listing 2: LaTeX code for Kalish-Montague-style proof

```
1   \[
2   \KMproof{
3     \cbblk{
4           \proofline {(((p\rightarrow q)\land(\neg r\rightarrow\neg q))\rightarrow(p\rightarrow r)
5     }{
6       \proofline {((p\rightarrow q)\land(\neg r\rightarrow\neg q))}{Supposition}
7       \cbblk{
8         \proofline {p\rightarrow r}{4--13 Conditionalization}
9       }{
10        \proofline {p}{Supposition}
11        \proofline {((p\rightarrow q)\land(\neg r\rightarrow\neg q))}{2 Repeat}
12        \proofline {(p\rightarrow q)}{5 Simplification}
13        \proofline {q}{4, 6 Modus Ponens}
14        \proofline {(\neg r\rightarrow\neg q)}{5 Simplification}
15        \cbblk{
16          \proofline {r}{10--13 Reductio ad Absurdum}
17        }{
18          \proofline {\neg r}{Supposition}
19          \proofline {(\neg r\rightarrow\neg q)}{8 Repeat}
20          \proofline {\neg q}{10, 11 Modus Ponens}
21          \proofline {q}{7 Repeat}
22        }
23      }
24    }
25  }
26  \]
```

Within the code to produce the Jaśkowski and the Kalish-Montague proofs we see two differences: First, the Jaśkowski proofs use the main control `\Jproof`, while the Kalish-Montague proofs employ `\KMproof`. Secondly, within each of these different proof controls are the commands for typesetting the conclusion: in the `\Jproof`, conclusions go *after* the supporting subproof, so we use `\cablk` for conclusion after block; in the `\KMproof` the conclusions come before the subproof so we use `\cbblk` for conclusion before block. And we have written the block structure before the conclusion in the Jaśkowski proof, while it is written after the conclusion in the Kalish-Montague proof.

On Klüwer's and Selinger's pages describing their two fitch.sty files, the

following argument is displayed:

1		$\exists x \forall y P(x,y)$	
2	v | u	$\forall y P(u,y)$	
3		$P(u,v)$	$\forall\mathbf{E}$, 2
4		$\exists x P(x,v)$	$\exists\mathbf{i}$, 3
5		$\exists x P(x,v)$	$\exists\mathbf{E}$, 1, 2–4
6		$\forall y \exists x P(x,y)$	$\forall\mathbf{I}$, 2–5

The idea is that u and v are *guards*, whose role is to prevent certain variables being imported into or exported out of the relevant subproof that they are guarding. A Jaśkowski style proof of this theorem is given in Figure **??** and is generated by Listing **??**.

Figure 1: A Jaśkowski-style proof with guarded variables

1.		$\exists x \forall y P(x,y)$	Premise
2.	v u	$\forall y P(u,y)$	Supposition
3.		$P(u,v)$	2, $\forall\boldsymbol{E}$
4.		$\exists x P(x,v)$	3, $\exists\boldsymbol{I}$
5.		$\exists x P(x,v)$	1,2–4, $\exists\boldsymbol{E}$
6.		$\forall y \exists x P(x,y)$	2–5, $\forall\boldsymbol{I}$

Listing 3: LaTeX code for Jaśkowski-style proof with guarded variables

```
\[
\Jproof
        {\proofline{\exists x\forall yP(x,y)}{Premise}
        \cablk[v]{
          \cablk[u]{
            \proofline{\forall yP(u,y)}{Supposition}
            \proofline{P(u,v)}{2, $\forall\bm{E}$}
            \proofline{\exists xP(x,v)}{3, $\exists\bm{I}$}
          }
          {\proofline{\exists xP(x,v)}{1,2--4, $\exists\bm{E}$}  }
        }
        {\proofline{\forall y\exists xP(x,y)}{2--5, $\forall\bm{I}$}
```

```
13            }
14          }
15  \]
```

Kalish-Montague's strategy of putting SHOW lines at the beginning of a subproof and of counting the (free) variables in that formula as if they actually occurred in the proof at the time one wishes to reiterate into the area beneath such an uncancelled SHOW (or to do $\exists \boldsymbol{E}$ by instantiating to variables that occur in the SHOW formula), makes it unnecessary to have explicit guards to protect $\forall \boldsymbol{I}$ and $\exists \boldsymbol{E}$, since these are the only reasons to have guards for variables – although one could indicate them using the present `\KMproof`, if one wished. (Another peculiarity of the Kalish-Montague system is that their $\exists$-elimination rule does not employ a subproof, but directly introduces a (completely) new variable into the proof, including being completely distinct from variables in SHOW lines – even uncancelled ones.) For these reasons we do not display the use of guards for variables in the Kalish-Montague system. But such guards are more logically useful in a Jaśkowski-style proof.

Another use of guards is in modal logic, where – depending on the particular modal system that we are providing a proof system for – certain formulas cannot be reiterated into a guarded-with-a-$\Box$ scope line. Here is an example in modal system S_4 (or any stronger one). A Jaśkowski style proof of the valid argument $\Box p \wedge \Box q \vdash \Box(p \wedge q)$ is given in Figure **??** and is generated by Listing **??**. It is followed by a Kalish-Montague style proof of the same argument in Figure **??** together with its code in Listing **??**.

Figure 2: A Jaśkowski-style proof with modal guards

1.		$\Box p \wedge \Box q$	premise
2.		$\Box p$	1,$\wedge \boldsymbol{E}$
3.		$\Box q$	1,$\wedge \boldsymbol{E}$
4.	$\Box$	$\Box p$	2, Reiterate
5.		$\Box q$	3, Reiterate
6.		p	4, $\Box \boldsymbol{E}$
7.		q	5, $\Box \boldsymbol{E}$
8.		$p \wedge q$	6,7, $\wedge \boldsymbol{I}$
9.		$\Box(p \wedge q)$	4–8, $\Box \boldsymbol{I}$

Listing 4: LaTeX code for Jaśkowski-style proof with modal guards

```
\[
\Jproof{
        \proofline{\bx p\land\bx q}{premise}
        \proofline{\bx p}{1,$\land\bm{E}$}
        \proofline{\bx q}{1,$\land\bm{E}$}
        \cablk[\bx]
             {\proofline{\bx p}{2, Reiterate}
               \proofline{\bx q}{3, Reiterate}
               \proofline{p}{4, $\bx\bm{E}$}
               \proofline{q}{5, $\bx\bm{E}$}
               \proofline{p\land q}{6,7, $\land\bm{I}$}
              }
             {\proofline{\bx(p\land q)}{4--8, $\bx\bm{I}$} }
         }
\]
```

Figure 3: A Kalish/Montague-style proof with modal guards

1.	~~*Show*~~ $\Box(p \wedge q)$	5, Direct Proof
2.	$\Box p \wedge \Box q$	Premise
3.	$\Box p$	2, $\wedge\boldsymbol{E}$
4.	$\Box q$	2, $\wedge\boldsymbol{E}$
5.	~~*Show*~~ $\Box(p \wedge q)$	6–10, $\Box\boldsymbol{I}$
6.	$\Box$ $\Box p$	3, Reiterate
7.	$\Box q$	4, Reiterate
8.	p	6, $\Box\boldsymbol{E}$
9.	q	7, $\Box\boldsymbol{E}$
10.	$(p \wedge q)$	8,9 $\wedge\boldsymbol{I}$

Listing 5: LaTeX code for Kalish-Montague-style proof with modal guards

```
\[
\KMproof{
```

```
3           \cbblk
4                   {\proofline{\bx (p \land q)}{5, Direct Proof} }
5                   {\proofline{\bx p \land \bx q}{Premise}
6                   \proofline{\bx p}{2, $\land\bm{E}$ }
7                   \proofline{\bx q}{2, $\land\bm{E}$ }
8                   \cbblk[\bx]
9                           { \proofline{\bx (p \land q)}{6--10, $\bx\bm{I}$} }
10                          { \proofline{\bx p}{3, Reiterate}
11                                  \proofline{\bx q}{4, Reiterate}
12                            \proofline{p}{6, $\bx\bm{E}$}
13                            \proofline{q}{7, $\bx\bm{E}$}
14                            \proofline{(p\land q)}{8,9 $\land\bm{I}$}   }
15                  }
16          }
17  \]
```

For further details on the history of natural deduction, including how it became the standard method in elementary logic textbooks, especially in the years 1950-1990 and beyond, see F.J. Pelletier (1999) "A Brief History of Natural Deduction" *History and Philosophy of Logic*, v. 20, pp. 1–31. Also discussed are the four main styles of representing natural deduction proofs: the Gentzen trees, the Jaśkowski-Fitch graphical (boxes) method, the Jaśkowski-Quine (1950) bookkeeping method, and the Suppes (1957) sequent natural deduction method. A discussion of how comparatively widespread these four methods have become is also indicated by a survey of many elementary natural deduction textbooks.