2.1 Inverse Hyperbolic Functions

A. Purpose

These subprograms compute the inverse hyperbolic functions.

B. Usage

B.1 Program Prototype, Single Precision

REAL X, U, SASINH, SACOSH, SATANH, SACTNH, SASECH, SACSCH

Assign a value to X and obtain the desired value of an inverse hyperbolic function using one of the following:

U = SASINH(X)	**U = SACOSH(X)**
U = SATANH(X)	**U = SACSCH(X)**
U = SASECH(X)	**U = SACTNH(X)**

where the functions SASINH, ..., SACTNH compute respectively, the inverse hyperbolic: sine, cosine, tangent, cosecant, secant, and cotangent.

B.2 Program Prototype, Double Precision

For double precision computation use the function names DASINH, DACOSH, DATANH, DACSCH, DASECH, and DACTNH and type the argument, function name and result double precision.

C. Example and Remarks

See DRSASINH and ODSASINH for an example of the usage of these subprograms.

D. Functional Description

D.1 Method

The basic formulas and valid argument domains are

$$
\begin{array}{ll}
\operatorname{asinh}(x) = \operatorname{sgn}(x)\ln(|x| + \sqrt{x^2+1}) & \text{all } x \\
\operatorname{acosh}(x) = \ln(x + \sqrt{x^2-1}) & x \ge 1 \\
\operatorname{atanh}(x) = \frac{1}{2}\operatorname{sgn}(x)\ln((1+|x|)/(1-|x|)) & |x| < 1 \\
\operatorname{acsch}(x) = \operatorname{asinh}(1/x) & x \ne 0 \\
\operatorname{asech}(x) = \operatorname{acosh}(1/x) & 0 < x \le 1 \\
\operatorname{actnh}(x) = \operatorname{atanh}(1/x) & |x| > 1
\end{array}
$$

To avoid unnecessary loss of relative accuracy as function values approach zero, a different method is used whenever the argument of the logarithm function would be in the range [1.0, 2.718]. In this range the argument for acosh or atanh is converted to an argument for asinh, and asinh is computed by argument reduction to the interval [0.0, 0.125326] followed by evaluation of its Taylor series. The number of terms needed in this series is determined by use of the System Parameters subprogram, Chapter 19.1. The prestored coefficients will support accuracy to about 25 significant decimal digits.

For large arguments the formulas involving x^2 are reformulated to avoid unnecessary overflow. Specifically, when $x > 10^{16}$ it is presumed that $x^2 \pm 1$ will not be distinguishable from x^2, and thus the formulas given above for asinh and acosh are replaced by

$$
\begin{aligned}
&\operatorname{asinh}(x) = \operatorname{sgn}(x)[\ln(2) + \ln(|x|)], \quad \text{and} \\
&\operatorname{acosh}(x) = \ln(2) + \ln(x)
\end{aligned}
$$

Let Ω denote the machine overflow limit and ρ denote the difference between 1.0 and the next smaller machine number. Define $a = \frac{1}{2}\ln(2/\rho)$ and $b = \ln(2\Omega)$. Then the ranges of the computed function values are

$$
\begin{array}{ll}
|\operatorname{asinh}(x)| < b & 0 \le \operatorname{acosh}(x) < b \\
|\operatorname{atanh}(x)| \le a & 0 < |\operatorname{acsch}(x)| < b \\
0 \le \operatorname{asech}(x) < b & 0 < |\operatorname{actnh}(x)| \le a
\end{array}
$$

D.2 Accuracy Tests

The single precision subprograms were tested on an IBM compatible PC by comparison with the double precision subprograms at 300,000 to 800,000 points in the domain of each function. Using $\rho = 2^{-23} \approx 0.119 \times 10^{-6}$, which is the relative precision of IEEE single precision arithmetic, these tests may be summarized as follows:

Function	**Argument Range**	**Max. Rel. Error**
SASINH	All x	0.9 ρ
SACOSH	[1.00, 1.21]	1.6 ρ
	$x \ge 1.21$	0.5 ρ
SATANH	[−0.44, 0.44]	1.3 ρ
	[0.44, 0.92]	1.3 ρ
	[0.92, 1.0]	0.5 ρ
SACSCH	$x \ne 0$	0.9 ρ
SASECH	[0.0, 0.24]	0.8 ρ
	[0.24, 0.68]	1.2 ρ
	[0.68, 0.88]	3.2 ρ
	[0.88, 1.]	989.1 ρ
SACTNH	[1.0, 1.16]	153.5 ρ
	[1.16, 2.2]	1.7 ρ
	$x \ge 2.2$	1.6 ρ

The instances of very large relative errors in this table are in regions where the slope of the graph of the function is becoming vertical. Note that relative errors may be significantly larger on machines that do not have properly rounded arithmetic.

The functions were tested using identities of the form $x - \sinh(\operatorname{asinh}(x)) = 0$. When scaled by the precision of the arithmetic used, the double precision and single precision functions had similar perfomance.

E. Error Procedures and Restrictions

If an argument is outside the valid domain, an error message will be issued, and the value zero will be returned. Error messages are processed using the subroutines of Chapter 19.2 with an error level of zero.

F. Supporting Information

The source language for these subroutines is ANSI Fortran 77.

All double precision entries are in the file DASINH, which also needs files: AMACH, DERM1, DERV1, ERFIN, and ERMSG.

All single precision entries are in the file SASINH, which also needs files: AMACH, ERFIN, ERMSG, SERM1, and SERV1.

Designed and programmed by C. Lawson and S. Chiu, JPL, 1983. Modified Nov., 1988 to use R1MACH and D1MACH.

Entries

DACOSH	DACSCH	SACOSH	SACSCH
DACTNH	DASECH	SACTNH	SASECH
DASINH	DATANH	SASINH	SATANH

DRSASINH

```
c       program DRSASINH
c>> 2001-05-22 DRSASINH Krogh Minor change for making .f90 version.
c>> 1996-06-26 DRSASINH Krogh Changed code to simplify conversion to C.
c>> 1996-05-28 DRSASINH Krogh Added external state. & moved up formats
c>> 1994-10-19 DRSASINH Krogh  Changes to use M77CON
c>> 1988-11-17 DRSASINH CLL
c--S replaces "?": DR?ASINH, ?ASINH, ?ACOSH, ?ATANH, ?ACTNH, ?ASECH, ?ACSCH
c
      integer I
      real             X1(3),X2(3)
      external SASINH,SACOSH,SATANH,SACTNH,SASECH,SACSCH
      real             SASINH,SACOSH,SATANH,SACTNH,SASECH,SACSCH
c
      data X1/0.1e0, 0.5e0, 0.9e0/
      data X2/1.1e0, 10.0e0, 100.0e0/
c
   10 format('      X',11X,'SASINH',10X,'SATANH',10X,
     *  'SASECH',10X,'SACSCH')
   20 format('     ',3('-'),10X,4(6('-'),10X))
   30 format('      X',11X,'SASINH',10X,'SACOSH',10X,
     *  'SACTNH',10X,'SACSCH')
c
      print 10
      print 20
      do 40 I = 1, 3
         print '(1X,F6.2,1X,4F16.6)', X1(I),SASINH(X1(I)),
     *      SATANH(X1(I)), SASECH(X1(I)),SACSCH(X1(I))
   40 continue
      print '('' ''/)'
      print 30
      print 20
      do 50 I = 1, 3
         print  '(1X,F6.2,1X,4F16.6)', X2(I),SASINH(X2(I)),
     *      SACOSH(X2(I)), SACTNH(X2(I)),SACSCH(X2(I))
   50 continue
      stop
      end
```

ODSASINH

```
      X           SASINH          SATANH          SASECH          SACSCH
     ---          ------          ------          ------          ------
    0.10        0.099834        0.100335        2.993223        2.998223
    0.50        0.481212        0.549306        1.316958        1.443635
    0.90        0.808867        1.472219        0.467145        0.957801


      X           SASINH          SACOSH          SACTNH          SACSCH
     ---          ------          ------          ------          ------
    1.10        0.950347        0.443568        1.522261        0.815609
   10.00        2.998223        2.993223        0.100335        0.099834
  100.00        5.298342        5.298292        0.010000        0.010000
```