2.17 Fresnel Integrals

A. Purpose

The subprograms described in this chapter compute the Fresnel Integrals

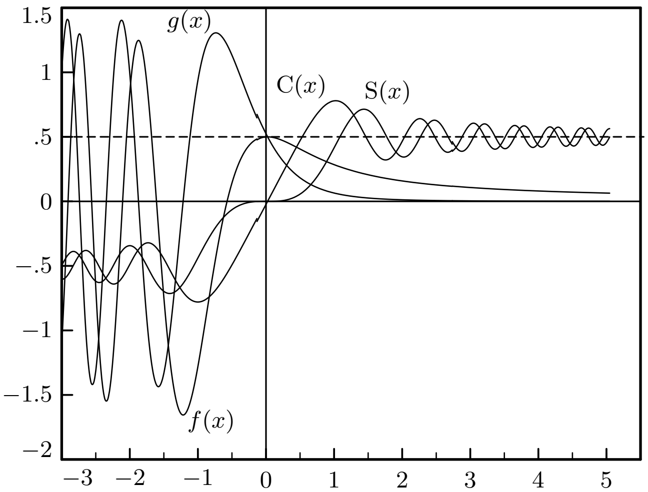
$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt \quad \text{and} \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt,$$

and the associated functions

$$f(x) = \left[\frac{1}{2} - S(x)\right] \cos\left(\frac{\pi}{2}x^2\right) - \left[\frac{1}{2} - C(x)\right] \sin\left(\frac{\pi}{2}x^2\right)$$

$$g(x) = \left[\frac{1}{2} - C(x)\right] \cos\left(\frac{\pi}{2}x^2\right) + \left[\frac{1}{2} - S(x)\right] \sin\left(\frac{\pi}{2}x^2\right)$$

as defined by Equations 7.3.1, 7.3.2, 7.3.5 and 7.3.6 in [1].



B. Usage

B.1 Program Prototype, Single Precision

To compute $C(x)$ use

REAL SFRENC, X, Y

$$\boxed{\mathbf{Y = SFRENC(X)}}$$

To compute $S(x)$ use

REAL SFRENS, X, Y

$$\boxed{\mathbf{Y = SFRENS(X)}}$$

To compute $f(x)$ use

REAL SFRENF, X, Y

$$\boxed{\mathbf{Y = SFRENF(X)}}$$

To compute $g(x)$ use

REAL SFRENG, X, Y

$$\boxed{\mathbf{Y = SFRENG(X)}}$$

B.2 Argument Definitions

X [in] The value at which the function is to be evaluated.

B.3 Modifications for Double Precision

Change the REAL type statements to double precision, and change the initial letter of the subprogram names from S to D. It is important that the subprogram names be explicitly typed.

C. Examples and Remarks

See DRSFRENL and ODSFRENL for an example of the usage of this subprogram.

There are no restrictions on the range of applicability of these functions. The accuracy of the trigonometric functions decreases, however, for large $|x|$. Thus evaluation of $C(x)$, $S(x)$, $f(-|x|)$ or $g(-|x|)$ for large $|x|$ will be less accurate than evaluation of $f(|x|)$ and $g(|x|)$ for the same value of x. When formulating an application, one should when possible use $C(x)$ and $S(x)$ when $|x| \leq 1.6$ (to achieve maximum efficiency). To achieve maximum accuracy and efficiency use $f(x)$ and $g(x)$ when $x > 1.6$, and avoid using $x < -1.6$.

D. Functional Description

The computer approximations for these functions use Chebyshev rational approximations developed by W. J. Cody, described in [2]. Cody provides approximations for $C(x)$ and $S(x)$ for $|x| \leq 1.6$, and for $f(x)$ and $g(x)$ for $x > 1.6$. The approximations for $f(x)$ and $g(x)$ for $x > 2.4$ have the same asymptotic form as the functions. We compute $f(x)$ and $g(x)$ from $S(x)$ and $C(x)$ when $|x| \leq 1.6$, and vice versa for $x > 1.6$. For $x < 0$ we use $C(-x) = -C(x)$, $S(-x) = -S(x)$, $g(-x) = \cos(\pi/2\ x^2) + \sin(\pi/2\ x^2) - g(x)$ and $f(-x) = \cos(\pi/2\ x^2) - \sin(\pi/2\ x^2) - f(x)$.

The approximations and programming were checked by comparing the double precision functions to an extended precision computation of $w(z)$, the Fadeeva function described in Chapter 2.16. Testing consisted of dividing several regions of the argument range into 200 equal-sized intervals, and selecting a point randomly in each interval. To test $f(x)$ and $g(x)$ when $50 < x < 1000$ we

Function	Argument Interval	Mean ULP	Max ULP	Mean REL	Max REL	Mean ABS	Max ABS
$C(x)$	[0..1.2]	0.57 ρ	2.18 ρ	0.40 ρ	1.29 ρ	0.19 ρ	0.83 ρ
	(1.2..1.6]	0.70 ρ	2.52 ρ	0.48 ρ	1.55 ρ	0.19 ρ	0.63 ρ
S(x)	[0..1.2]	0.74 ρ	2.42 ρ	0.52 ρ	1.39 ρ	0.09 ρ	0.65 ρ
	(1.2..1.6]	0.75 ρ	2.20 ρ	0.55 ρ	1.55 ρ	0.38 ρ	1.10 ρ
$f(x)$	(1.6..1.9]	0.51 ρ	1.50 ρ	0.36 ρ	1.10 ρ	0.06 ρ	0.19 ρ
	(1.9..2.4]	0.30 ρ	0.96 ρ	0.26 ρ	0.77 ρ	0.04 ρ	0.12 ρ
	(2.4..6.0]	0.43 ρ	1.15 ρ	0.29 ρ	0.80 ρ	0.02 ρ	0.07 ρ
	(6.0..50.0]	0.45 ρ	1.05 ρ	0.31 ρ	0.71 ρ	0.00 ρ	0.03 ρ
	(50..1000]	0.39 ρ	1.02 ρ	0.27 ρ	0.72 ρ	0.00 ρ	4E−3 ρ
$g(x)$	(1.6..1.9]	0.53 ρ	1.92 ρ	0.38 ρ	1.11 ρ	0.01 ρ	0.02 ρ
	(1.9..2.4]	1.06 ρ	3.43 ρ	0.75 ρ	1.93 ρ	0.01 ρ	0.02 ρ
	(2.4..6.0]	1.51 ρ	4.04 ρ	1.01 ρ	2.61 ρ	0.00 ρ	0.01 ρ
	(6.0..50.0]	1.40 ρ	3.62 ρ	0.97 ρ	2.11 ρ	0.00 ρ	3E−4 ρ
	(50..1000]	1.09 ρ	2.40 ρ	0.75 ρ	1.50 ρ	0.00 ρ	2E−7 ρ

divided the range $10^{-3} < 1/x < .02$ into 200 equal subranges. In each interval we report the error in units of the last position of the test value in the column headed ULP, the error relative to the true value in the column headed REL, and the absolute error in the column headed ABS. The quantity ρ is the round off level, that is, the difference between 1.0 and the next representable number, which is provided by D1MACH(4) (Chapter 19.1). For IEEE arithmetic, $\rho \approx 2.22\text{E}{-16}$ in double precision. The results are summarized above.

Cody's testing of the approximations, as described in [2], indicates a relative accuracy in the approximations of 15 to 18 digits, so one should not expect to achieve more accuracy simply by carrying out the calculations using more precision, as, for example, by using double precision on a Cray computer.

The errors in $f(x)$ and $g(x)$ decrease as x increases in the range $6 < x \leq 1000$. The approximations for $f(x)$ and $g(x)$ have the same asymptotic form as the functions when $x > 2.4$, and therefore they become more accurate as x increases. For IEEE format double precision arithmetic, the approximation for $f(x)$ is identical to the asymptotic expansion when $x > 29$, and the approximation for $g(x)$ is identical to the asymptotic expansion when $x > 14$.

We verified correct programming of $f(x)$ and $g(x)$ for $|x| \leq 1.6$, and for $C(x)$ and $S(x)$ for $x > 1.6$, by comparing results to values in table 7.7 in [1]. Extensive accuracy testing would simply have validated the trigonometric function routines.

References

1. Milton Abramowitz and Irene A. Stegun, **Handbook of Mathematical Functions**, *Applied Mathematics Series 55*, National Bureau of Standards (1966).

2. W. J. Cody, *Chebyshev approximations for the Fresnel integrals*, **Math. of Comp. 22**, 102 (April 1968) 450–453 (plus microfiche supplement).

E. Error Procedures and Restrictions

There are no restrictions on the argument range for these functions; they do not announce any errors.

F. Supporting Information

The source language is ANSI Fortran 77.

Entry	Required Files
DFRENC	AMACH, DFRENL
DFRENF	AMACH, DFRENL
DFRENG	AMACH, DFRENL
DFRENS	AMACH, DFRENL
SFRENC	AMACH, SFRENL
SFRENF	AMACH, SFRENL
SFRENG	AMACH, SFRENL
SFRENS	AMACH, SFRENL

Subprograms designed and developed by W. V. Snyder, JPL, 1992.

DRSFRENL

```
c      program DRSFRENL
c>> 2001-05-25 DRSFRENL Krogh -- Added comma to format.
c>> 1996-06-18 DRSFRENL Krogh Minor changes in formats for C conversion.
c>> 1996-01-29 DRSFRENL WV Snyder Corrected formats
c>> 1994-10-19 DRSFRENL Krogh  Changes to use M77CON
c>> 1993-02-25 DRSFRENL CLL. Minor edits.  Deleted Format statements.
c>> 1992-03-18 DRSFRENL WV Snyder Corrected failure to call SFRENS
c>> 1992-03-18 DRSFRENL CLL Minor edits.
c>> 1992-02-03 DRSFRENL WV Snyder JPL Original code.
c
c      Demonstration driver for Fresnel Integrals functions.
c
c      ------------------------------------------------------------------
c--S replaces "?": DR?FRENL, ?FRENC, ?FRENS, ?FRENG, ?FRENF
c      ------------------------------------------------------------------
      real             X, YC, YF, YG, YS
      real             SFRENC, SFRENF, SFRENG, SFRENS
      external SFRENC, SFRENF, SFRENG, SFRENS
      integer I
c
      print '(11x,''X'',9x,''C(x)'',11x,''S(x)'',11x,''g(x)'',11x,
     *   ''f(x)'')'
      do 30 I = -12, 12
         X = 0.5 * I
         YC = SFRENC(X)
         YS = SFRENS(X)
         YG = SFRENG(X)
         YF = SFRENF(X)
         print '(1p,5e15.07)', X, YC, YS, YG, YF
30    continue
      stop
      end
```

ODSFRENL

X	C(x)	S(x)	g(x)	f(x)
−6.0000000E+00	−4.9953148E−01	−4.4696075E−01	9.9953145E−01	9.4696075E−01
−5.5000000E+00	−4.7842142E−01	−5.5368412E−01	−1.3071709E+00	−5.9905142E−01
−5.0000000E+00	−5.6363118E−01	−4.9919137E−01	9.9919140E−01	−1.0636312E+00
−4.5000000E+00	−5.2602589E−01	−4.3427297E−01	1.3054551E+00	4.7051251E−01
−4.0000000E+00	−4.9842602E−01	−4.2051578E−01	9.9842602E−01	9.2051578E−01
−3.5000000E+00	−5.3257245E−01	−4.1524801E−01	1.3042227E+00	4.5043051E−01
−3.0000000E+00	−6.0572076E−01	−4.9631301E−01	9.9631298E−01	−1.1057208E+00
−2.5000000E+00	−4.5741299E−01	−6.1918175E−01	−1.3128265E+00	−6.6760296E−01
−2.0000000E+00	−4.8825341E−01	−3.4341568E−01	9.8825341E−01	8.4341568E−01
−1.5000000E+00	−4.4526112E−01	−6.9750500E−01	−1.3315728E+00	−7.4461448E−01
−1.0000000E+00	−7.7989340E−01	−4.3825918E−01	9.3825918E−01	−1.2798934E+00
−5.0000000E−01	−4.9234423E−01	−6.4732432E−02	1.1329203E+00	1.4199099E−01
0.0000000E+00	0.0000000E+00	0.0000000E+00	5.0000000E−01	5.0000000E−01
5.0000000E−01	4.9234423E−01	6.4732432E−02	1.7364271E−01	3.9920503E−01
1.0000000E+00	7.7989340E−01	4.3825918E−01	6.1740816E−02	2.7989340E−01
1.5000000E+00	4.4526112E−01	6.9750500E−01	2.5009766E−02	2.0341848E−01
2.0000000E+00	4.8825341E−01	3.4341568E−01	1.1746596E−02	1.5658432E−01
2.5000000E+00	4.5741299E−01	6.1918175E−01	6.2636342E−03	1.2640692E−01
3.0000000E+00	6.0572076E−01	4.9631301E−01	3.6870006E−03	1.0572079E−01
3.5000000E+00	5.3257245E−01	4.1524801E−01	2.3401754E−03	9.0765551E−02
4.0000000E+00	4.9842602E−01	4.2051578E−01	1.5739668E−03	7.9484239E−02
4.5000000E+00	5.2602589E−01	4.3427297E−01	1.1078332E−03	7.0683539E−02
5.0000000E+00	5.6363118E−01	4.9919137E−01	8.0861803E−04	6.3631192E−02
5.5000000E+00	4.7842142E−01	5.5368412E−01	6.0798827E−04	5.7855371E−02

6.0000000E+00	4.9953148E−01	4.4696075E−01	4.6853212E−04	5.3039238E−02