2.19 Incomplete Gamma Function Ratio

A. Purpose

The subroutines described here compute the incomplete gamma function ratios, defined by the equations

$$P(a,x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt$$

$$Q(a,x) = \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt$$

$$P(a,x) + Q(a,x) = 1.$$

These functions are related to several other functions, *viz.*

Chi-squared distribution and complementary Chi-squared distribution (see Chapter 15.3):

$$P(\chi^2|\nu) = P(\nu/2, \chi^2/2), \quad Q(\chi^2|\nu) = Q(\nu/2, \chi^2/2).$$

The incomplete gamma functions, when $a > 0$:

$$\gamma(a,x) = \Gamma(a)P(a,x), \quad \Gamma(a,x) = \Gamma(a)Q(a,x).$$

Tricomi's incomplete gamma function:

$$\gamma^*(a,x) = x^{-a}P(a,x).$$

When a is a small integer or half-integer, incomplete gamma functions reduce to simpler special functions, or elementary functions, which ought to, and sometimes must be, used in place of $P(a,x)$ or $Q(a,x)$ for computation:

$\Gamma(0,x) = E_1(x)$ (See Chapter 2.10.) Notice this cannot be computed as $\Gamma(0)Q(0,x)$.

$P(\frac{1}{2},x) = \mathrm{erf}(\sqrt{x})$, $Q(\frac{1}{2},x) = \mathrm{erfc}(\sqrt{x})$ See Chapters 2.2 and 2.16; SGAMI uses this relation.)

$Q(1,x) = \Gamma(1,x) = e^{-x}$.

The recurrence relation $P(a+1,x) = P(a,x) - x^a e^{-x}/\Gamma(a+1)$ may be used with the above relations, but errors will accumulate if a is large.

Consult [1] for additional relations and mathematical properties.

B. Usage

B.1 Program Prototype, Single Precision

REAL A, X, P, Q

INTEGER IERR

Assign values to A and X, and obtain P = $P(a,x)$ and Q = $Q(a,x)$ by using

CALL SGAMI (A, X, P, Q, IERR)

B.2 Argument Definitions

A [in] The parameter of the incomplete gamma function ratio, a above.

X [in] The argument of the incomplete gamma function ratio, x above. Require ($x > 0$ and $a \geq 0$) or ($x \geq 0$ and $a > 0$).

P [out] The incomplete gamma function ratio P(a,x).

Q [out] The complementary incomplete gamma function ratio Q(a,x).

IERR [out] is a status flag that is zero upon return if computation was successful. See Section E for description of nonzero values.

B.3 Program Prototype, Single Precision, Specify Options

REAL PTOL, QTOL, XERR

INTEGER MSGOFF

Assign values to PTOL, QTOL, XERR and MSGOFF, and specify options for SGAMI by using

CALL SGAMIK (PTOL, QTOL, XERR, MSGOFF)

B.4 Argument Definitions

PTOL, QTOL [in] Error tolerances for P and Q, respectively. When positive, they indicate relative error tolerances. When negative, they indicate the negative of an absolute tolerance. When zero, they indicate the default, a relative error tolerance equal to four times the round-off level, should be used.

XERR [in] If non-negative, XERR provides the estimated relative error in X. If negative, XERR indicates the default error estimate for X, the round-off level, should be used.

MSGOFF [in] MSGOFF is added onto the error message level before an error message is produced by using the error message processor described in Chapter 19.2.

If SGAMIK is not called, the effect is as though CALL SGAMIK (0.0, 0.0, −1.0, 0) had been executed.

B.5 Program Prototype, Single Precision, Determine Error

REAL PQERR

Retrieve the absolute error committed by the last call to SGAMI by using

CALL SGAMIE (PQERR)

B.6 Argument Definitions

PQERR [out] reports the absolute error committed by the last call to SGAMI. If SGAMI has not been called, PQERR is returned with the value −1.

B.7 Modifications for Double Precision

Change the REAL statement to DOUBLE PRECISION and change the prefix of the subroutine names from SGAMI to DGAMI.

C. Examples and Remarks

DRDGAMI uses DGAMI to evaluate the identities $P(\frac{1}{2}, x) = \operatorname{erf}\sqrt{x}$ and $Q(\frac{1}{2}, x) = \operatorname{erfc}\sqrt{x}$ for several values of x. Results are in ODDGAMI.

SGAMI computes an estimate for the absolute error = (relative error in X) × X × |∂P/∂X| = (relative error in X) × X × |∂Q/∂X|. When A and X are large and A ≈ X, X |∂P/∂X| ≈ A |∂P/∂A| ≈ $(\mathrm{A}/2\pi)^{\frac{1}{2}}$ and $P(\mathrm{A}, \mathrm{X}) \approx Q(\mathrm{A}, \mathrm{X}) \approx 0.5$. Thus as A and X increase the relative error in either of them is magnified in the result in proportion to the square root of their values. When A or X is exactly zero, X $\partial P/\partial$X = A $\partial P/\partial$A = 0. Otherwise, it is more expensive to estimate A $\partial P/\partial$A than to estimate P(A, X) (A Meijer G function is required), but A $\partial P/\partial$A is usually less than MAX(1.0, X $\partial P/\partial$X). In many applications, furthermore, A is known exactly, and is exactly representable, so A $\partial P/\partial$A × (relative error in A) is identically zero.

When A and X are both large and nearly equal, the relative error in P and Q is approximately $(\mathrm{A}/2\pi)^{\frac{1}{2}} \times \max(\varepsilon_A, \varepsilon_X)$, where ε_A and ε_X are the relative errors in A and X, respectively. If $\max(\varepsilon_A, \varepsilon_X) \approx \rho$, where ρ is the round-off level for the arithmetic, the relative error in P or Q is approximately 1.0, that is, no precision remains, when $\mathrm{A} > 2\pi\rho^{-2}$.

The function P (Q) decreases approximately as $e^{x-a}(e^{a-x})$. A first-order approximation to $e^{|x-a|}$ shows that the relative error in P (Q) is approximately equal to the absolute error in $|x - a|$ when $|x - a|$ is large. The representation of $P(Q)$ underflows when $|x - a|$ is greater than approximately $\ln(\Omega)$, where Ω is the largest representable floating point number.

D. Functional Description

The incomplete gamma function ratios enjoy the relations $0 \le P(a, x), Q(a, x) \le 1$ and $P(a, x) + Q(a, x) = 1$. Furthermore, $P(a, x) \to 1$ and $Q(a, x) \to 0$ when $a \to 0$ with $x > 0$. SGAMI and DGAMI define $P(0, x) = 1$ and $Q(0, x) = 0$ when $x > 0$. We also have $P(a, x) \to \frac{1}{2} + \frac{1}{2}\operatorname{erf} y$, $Q(a, x) \to \frac{1}{2}\operatorname{erfc} y$, $P(a, a) \to \frac{1}{2}$, $Q(a, a) \to \frac{1}{2}$ when $a \to \infty$, x fixed, $y = \sqrt{2}(x - a)/2\sqrt{a}$. $P(a, x)$ and $Q(a, x)$ are not defined for $a < 0$ or $x < 0$, but Tricomi's entire incomplete gamma function $\gamma^*(a, x)$ is a single-valued analytic function of x and a, possessing no finite singularities.

$P(a, x)$ enjoys the recurrence relation $P(a + 1, x) = P(a, x) - x^a e^{-x}/\Gamma(a + 1)$, from which one can compute $P(n+\frac{1}{2}, z)$ for small integer n and arbitrary complex z by using CWOFZ from Chapter 2.16. $Q(n, z)$ is elementary, being a polynomial of degree n in z multiplied by e^{-z}. See Sections 6.5 and 26.4 of [1] for further properties of the incomplete gamma function ratios.

The computational methods used in SGAMI and DGAMI are described in [2]. Methods include continued fractions, series expansions, recurrences, and uniform asymptotic expansions.

Accuracy Testing

The precision of SGAMI was assessed by comparing to DGAMI on an IBM compatible PC with an 80486 processor, using IEEE arithmetic for which the round-off level $\rho \approx 1.19 \times 10^{-7}$. The results are shown in the following table. The quantity R is the relative error in units of ρ; the quantity B is the error relative to the achievable precision in units of ρ, which takes into account the derivative of the function: Where the derivative has large magnitude, a small perturbation of the arguments causes a large perturbation of the result, so achieving precise results is inherently difficult. The column labeled "% 1 bit" is the percentage of samples for which no more than 1 bit was incorrect, in the relative error measure. For the case of the range $(1, 10^7) \times (1, 10^7)$, $(0, 7) \times (0, 7)$ was divided into 288×288 equal subregions, and a point (a, x) was randomly selected in each subregion. The function was then evaluated with arguments 10^a and 10^x. All other ranges were divided into 288×288 equal subregions, and the function was evaluated at a point (a,x) randomly selected in each subregion. The greatest possible number of samples in each range was 82944; fewer samples were used in some ranges if DGAMI estimated it was unable to produce answers more precise than single precision, but no range was tested using fewer than 79918 samples.

range	Max |R|	Max |B|	% 1 bit
$(0,100)\times(0,100)$	82.98	5.91	65.4
$(1,10^7)\times(1,10^7)$	178.23	1.72	97.0
$(0,1)\times(0,1)$	3.11	1.53	98.1
$(1,10)\times(1,10)$	6.98	2.13	95.7
$(1,10)\times(0,1)$	16.80	0.96	61.6
$(0,1)\times(1,10)$	2.09	2.09	99.9

The version of DGAMI used for reference was developed and tested by A. H. Morris, and described in [2].

The accuracy of Morris's version of DGAMI, and of the version of DGAMI incorporated into MATH77, were assessed by checking their results against 50 digit computations carried out by the computer algebra program *Maple*, at 30 points in the range $(1,100)\times(1,100)$ where SGAMI committed the greatest relative error. In all cases, the results were correct to more than 14 digits. The precision of DGAMI was not as rigorously assessed as that for SGAMI.

References

1. Milton Abramowitz and Irene A. Stegun, **Handbook of Mathematical Functions**, *Applied Mathematics Series 55*, National Bureau of Standards (1966) Chapter 6, 253–293.
2. Armido R. DiDonato and Alfred H. Morris, *Computation of the incomplete gamma function ratios and their inverse*, **ACM Trans. on Math. Software 12**, 4 (Dec. 1986) 377–393.

E. Error Procedures and Restrictions

If the error tolerances are not satisfied, $P(a,x)$ and $Q(a,x)$ will be computed, but an error message will be issued by the error message processor of Chapter 19.2, with LEVEL = 2 + MSGOFF, where MSGOFF is zero unless specified by a call to SGAMIK at some time before calling SGAMI. If error termination is suppressed by providing a sufficiently small value of MSGOFF (say -2), or by calling ERMSET, IERR is set to 2.

If SGAMI or DGAMI is called with A = X = 0, neither $P(a,x)$ nor $Q(a,x)$ is defined, and an error message is issued by the error message processor of Chapter 19.2, with LEVEL = 2 + MSGOFF. If error termination is suppressed, IERR is set to 3 and P is set to 3.0.

If SGAMI or DGAMI is called with at least one of A or X negative, neither $P(a,x)$ nor $Q(a,x)$ is defined, and an error message is issued by the error message processor as above. If error termination is suppressed, IERR is set to 4 and P is set to 4.0.

F. Supporting Information

All program units are written in ANSI Standard Fortran 77. The program units SGAMI, SGAMIB, SGAMIE and SGAMIK communicate by way of a common block /SGAMIC/. The program units DGAMI, DGAMIB, DGAMIE and DGAMIK communicate by way of a common block /DGAMIC/.

Designed and programmed by Armido DiDonato and Alfred Morris, Naval Surface Warfare Center, Dahlgren, VA 22448–5000, 1986. Revised and adapted to Math 77 by W. V. Snyder, 1993.

Entry	Required Files
DGAMI	AMACH, DCSEVL, DERF, DERM1, DERV1, DGAM1, DGAMI, DGAMMA, DGR17, DGR29, DINITS, DRCOMP, DREXP, DRLOG, DXPARG, ERFIN, ERMSG, IERM1, IERV1
DGAMIE	AMACH, DCSEVL, DERF, DERM1, DERV1, DGAM1, DGAMI, DGAMMA, DGR17, DGR29, DINITS, DRCOMP, DREXP, DRLOG, DXPARG, ERFIN, ERMSG, IERM1, IERV1
DGAMIK	AMACH, DCSEVL, DERF, DERM1, DERV1, DGAM1, DGAMI, DGAMMA, DGR17, DGR29, DINITS, DRCOMP, DREXP, DRLOG, DXPARG, ERFIN, ERMSG, IERM1, IERV1
SGAMI	AMACH, ERFIN, ERMSG, IERM1, IERV1, SCSEVL, SERF, SERM1, SERV1, SGAM1, SGAMI, SGAMMA, SINITS, SRCOMP, SREXP, SRLOG, SXPARG
SGAMIE	AMACH, ERFIN, ERMSG, IERM1, IERV1, SCSEVL, SERF, SERM1, SERV1, SGAM1, SGAMI, SGAMMA, SINITS, SRCOMP, SREXP, SRLOG, SXPARG
SGAMIK	AMACH, ERFIN, ERMSG, IERM1, IERV1, SCSEVL, SERF, SERM1, SERV1, SGAM1, SGAMI, SGAMMA, SINITS, SRCOMP, SREXP, SRLOG, SXPARG

DRDGAMI

```
      program drdgami
c>>   2001-05-25 DRDGAMI Krogh Minor change for making .f90 version.
c>>   1996-05-28 DRDGAMI Krogh  Changed Fortran 90 code.
c>>   1994-10-19 DRDGAMI Krogh  Changes to use M77CON
c>>   1994-08-15 DRDGAMI WV Snyder JPL set up for chgtyp
c>>   1993-08-03 DRDGAMI WV Snyder JPL Original code
c--D replaces "?": DR?GAMI, ?ERF, ?ERFC, ?GAMI, ?GAMIK
c
c     Demo driver for incomplete gamma function.
c     Evaluate the identity P(1/2,x) = erf(sqrt(x)) and Q(1/2,x) =
c     erfc(sqrt(x)) for several values of x.
c
      double precision A, X, P, Q, DERF, DERFC, S, SC, SX
      external DERF, DERFC
      integer I, IND
      double precision XS(5)
      data XS /1.0d0, 2.0d0, 3.0d0, 4.0d0, 5.0d0/
c
10    format ('            X          P(1/2,X)        ERF(SQRT(X))    ',
     1        '    Q(1/2,X)        ERFC(SQRT(X))')
20    format (1p,5e16.8)
      print 10
      a = 0.5d0
      ind = 0
      call dgamik (0.0d0, 0.0d0, 0.0d0, 0)
      do 30 i = 1, 5
         x = xs(i)
         call dgami (a, x, p, q, ind)
         sx = sqrt(x)
         s = derf(sx)
         sc = derfc(sx)
         print 20, x, p, s, q, sc
30    continue
      stop
      end
```

ODDGAMI

X	P(1/2,X)	ERF(SQRT(X))	Q(1/2,X)	ERFC(SQRT(X))
1.00000000E+00	8.42700793E−01	8.42700793E−01	1.57299207E−01	1.57299207E−01
2.00000000E+00	9.54499736E−01	9.54499736E−01	4.55002639E−02	4.55002639E−02
3.00000000E+00	9.85694122E−01	9.85694122E−01	1.43058784E−02	1.43058784E−02
4.00000000E+00	9.95322265E−01	9.95322265E−01	4.67773498E−03	4.67773498E−03
5.00000000E+00	9.98434598E−01	9.98434598E−01	1.56540226E−03	1.56540226E−03